February 28, 1878.

Sir JOSEPH HOOKER, K.C.S.I., President, in the Chair.

The Right Hon. Sir William Henry Gregory was admitted into the Society.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read :—

I. "On certain Definite Integrals." By W. H. L. RUSSELL, F.R.S. Received January 10, 1878.

The integrals in the three preceding papers may nearly all be included under the following general theorem :—

Let $\phi(x)=u_0+u_1x+u_2x^r+\ldots u_rx^r+\ldots$ which series is, of course, supposed to be convergent, and let Π be a general functional symbol, such that

$$\Pi\phi(x)=u_0\chi(0), \quad +u_1\chi(1)\, x+u_2\chi(2)\, x^2+\ldots$$

and let $\chi(\theta)=\int \mathrm{U}d\theta$, $\chi(1)=\int \mathrm{UV}d\theta \ldots \chi(r)=\int \mathrm{UV}^r d\theta \ldots$ the integrals being supposed to be taken within certain limits: then

$$(62.) \qquad \int \mathrm{U}\phi(\mathrm{V}x)d\theta=\Pi\phi(x).$$

The following integrals require for the most part other methods to determine them:

$$(63.) \quad \int_0^\infty \frac{dx}{(1+x^2)^2(1-2a\cos x+a^2)}=\frac{\pi}{4}\,\frac{1}{1-a^2}\left\{\frac{\epsilon+a}{\epsilon-a}+\frac{2a\epsilon}{(\epsilon-a)^2}\right\}.$$

More generally we may obtain

$$(64.) \quad \int_0^\infty \frac{dx}{(1+x^2)^r(1-2a\cos x+a^2)}. \qquad (65.) \quad \int_0^\infty \frac{dx\,.\,x\sin ra}{(1+x^2)^r(1-2a\cos x+a^2)}.$$

$$(66.) \quad \int_0^\infty \frac{dx}{(1+x^2)^r}\cdot\frac{\sin ax}{\sin bx}. \qquad (67.) \quad \int_0^\infty \frac{dx}{(1+x^2)^r}\cdot\frac{\cos ax}{\cos bx}.$$

$$(68.) \quad \int_0^\infty \frac{dx}{1+x^{2r}}\cdot\frac{\sin ax}{\sin bx}. \qquad (69.) \quad \int_0^\infty \frac{dx}{1+x^{2r}}\cdot\frac{\cos ax}{\cos bx}.$$

$$(70.) \quad \int_0^\infty \frac{(x-a^2x)dx}{((1+a+a^2)\sin ax-a\sin 3ax)\,(1+x^2)^r}.$$

$$(71.) \quad \int_0^\infty \frac{(1-a^2)dx}{((1+a+a^2)\cos ax+a\cos 3ax)(1+x^2)^r}.$$

The integral (68.) calls for some particular remarks. Since by the ordinary rules:

$$\sin\theta+\sin 3\theta+\ \ldots\ \sin(2r+1)\theta=\frac{2\sin\theta-\sin(2r+3)\theta+\sin(2r+1)\theta}{2(1-\cos 2\theta)}$$

whence also:

$$\int_0^\infty dx,\ \sin ax\ .\ \frac{2\sin bx-\sin(2n+3)bx+\sin(2n+1)bx}{(1+x^{2r})(1-\cos 2bx)}$$

$$=2\int_0^\infty\frac{dx\sin ax\sin bx}{1+x^{2r}}+2\int_0^\infty\frac{dx\sin ax\sin 3bx}{1+x^{2r}}$$

$$+\ \ldots\ +2\int_0^\infty\frac{dx\sin ax\sin(2n+1)bx}{1+x^{2r}}.$$

Now
$$2\sin ax\sin bx=\cos(b-a)x-\cos(b+a)x$$
$$2\sin ax\sin 3bx=\cos(3b-a)x-\cos(3b+a)x$$
$$\ldots\ =\ \ldots$$
$$\int_0^\omega\frac{dx\cos mx}{1+x^{2r}}=C_1\epsilon^{-c_1m}+C_2\epsilon^{-c_2m}\ \ldots\ +C_r\epsilon^{-c_sm}.$$

Where $-c_1$, $-c_2$, $-c_3$ . . $-c_s$ are all the roots of the equation $x^{2r}+1=0$ which have a negative modulus, and C_1, C_2, C_3 . . . C_s are certain constants whose values have been assigned by Poisson: hence

$$\int_0^\omega dx\ .\sin ax\ .\ \frac{2\sin bx-\sin(2n+3)bx+\sin(2n+1)bx}{(1+x^{2r})(1-\cos 2bx)}=C_1\phi(c_1)+C_2\phi(c_2)$$

$$+\ \ldots\ +C_s\phi(c_s)_1,\text{ when }\phi(c)=(\epsilon^{cx}-\epsilon^{-ca})\ .\ \frac{\epsilon^{-cb}-\epsilon^{-c(2n+3)b}}{1-\epsilon^{-2cb}}.$$

Now let (n) increase without limit, then

$$\sin(2n+3)bx=\sin 2n\left(1+\frac{3}{2n}\right)bx=\sin 2nx.$$

$$\sin(2n+1)bx=\sin 2n\left(1+\frac{1}{2n}\right)bx=\sin 2nx.$$

Hence the integral in the second member vanishes, and

$$\int_0^\omega\frac{dx}{1+x^{2r}}\cdot\frac{\sin ax}{\sin bx}=C_1\frac{(\epsilon^{c_1a}-\epsilon^{-c_1a})\epsilon^{c_1b}}{\epsilon^{2c_1b}-1}+C_2\frac{(\epsilon^{c_2a}-\epsilon^{-c_2a})\epsilon^{c_2b}}{\epsilon^{2c_2b}-1}+\ \ldots\ (s\text{ terms}).$$

It is scarcely necessary to observe, that this process implies that (a) must be less than (b), otherwise the integral $\int_0^\omega\frac{\cos mx\ .\ dx}{1+x^{2r}}$ as here used will be discontinuous.

(72.) $$\int_0^\pi\frac{d\theta\sin^{2r}\theta}{\epsilon^{\cos\theta}+1}=\frac{(2r-1)(2r-3\ ..}{2r(2r-2)\ \ldots}\ \frac{1}{2}\cdot\frac{\pi}{2}.$$

(73.) $$\int_0^\pi\frac{d\theta}{(1+c^2\sin^2\theta)(\epsilon^{\cos\theta}+1)}=\frac{\pi}{2\sqrt{1+c^2}}.$$

(74.) $\displaystyle\int_0^\pi \frac{d\theta\phi(\sin^2\theta)}{\epsilon^{\cos\theta}+1}=\frac{1}{2}\int_0^\pi d\theta\phi(\sin^2\theta).$

Now let us define four quantities, ρ_1, ρ_2, μ_1, μ_2, thus

$$\rho_1=\frac{1}{\sqrt{2}}\left\{\sqrt{1+a^2+a^4}+\left(1+\frac{a^2}{2}\right)\right\}^{\frac{1}{2}},$$

$$\rho_2=\frac{1}{\sqrt{2}}\left\{\sqrt{1+a^2+a^4}-\left(1+\frac{a^2}{2}\right)\right\}^{\frac{1}{2}},$$

$$\mu_1=\rho_2\sqrt{3}-\rho_1+1,\quad \mu_2=\rho_2+\rho_1\sqrt{3}-\sqrt{3},$$

then we shall have

(75.) $\displaystyle\int_1^\omega \frac{x^3+a^3}{x^3-a^3}\frac{dx}{\sqrt{x^2-1}}=\pi\{\sin^{-1}a+\sin^{-1}(\rho_2\sqrt{3}-\rho_1)a\}.$

(76.) $$\int_0^\pi \frac{d\theta\cos r\theta}{1+a^3\cos^3\theta}=\frac{\pi}{3\sqrt{1-a^2}}\left\{\frac{\sqrt{1-a^2}-1}{a}\right\}^r+\frac{2\pi}{3(2a)^r(\rho_1^2+\rho_2^2)}$$
$$\left\{\rho_1\left(\mu_1^r-r\frac{r-1}{2}\mu_1^{r-2}\mu_2^r+r\,.\,\frac{r-1}{2}\,.\,\frac{r-2}{3}\,.\,\frac{r-3}{4}\mu_1^{r-4}\mu_2^4-\,..\right)\right.$$
$$\left.-\rho_2\left(r\mu_1^{r-1}\mu_2-r\,.\,\frac{r-1}{2}\,.\,\frac{r-2}{3}\mu_1^{r-3}\mu_2^3+\,..\right)\right\}.$$

(77.) $$\int_0^\pi \frac{d\theta\epsilon^\theta\cos\theta}{1+a^3\cos^3\theta}=\frac{\pi}{3\sqrt{1-a^2}}\epsilon^{\frac{\sqrt{1-a^2}-1}{a}}$$
$$+\frac{2\pi}{3(\rho_1^2+\rho_2^2)}\epsilon^{\frac{\mu_1}{2a}}\left(\rho_1\cos\frac{\mu_2}{2a}-\rho_2\sin\frac{\mu_2}{2a}\right).$$

(78.) $$\int_0^\pi d\theta\log_\epsilon(1+a^3\cos^3\theta)=\pi\log_\epsilon\frac{1+\sqrt{1-a^2}}{8}$$
$$+\pi\log_\epsilon(1+\sqrt{(a^2+2\sqrt{1+a^2+a^4}+2)}+\sqrt{1+a^2+a^4}).$$

(79.) $$\int_0^\pi d\theta\cos r\theta\log_\epsilon(1+a^3\cos^3\theta)=\frac{1}{r}\left\{\frac{\sqrt{1-a^2}-1}{a}\right\}^r+$$
$$\frac{2}{r(2a)^r}\left(\mu_1^r-r\,.\,\frac{r-1}{2}\mu_1^{r-2}\mu_2^r+r\,.\,\frac{r-1}{2}\,.\,\frac{r-2}{3}\,.\,\frac{r-3}{4}\mu_1^{r-4}\mu_2^2-\,...\right).$$

(80.) From (79.) we can immediately deduce

$$\int_0^\pi \frac{m^2+m\cos\theta}{1+2m\cos\theta+m^2}d\theta\log_\epsilon(1+a^3\cos^3\theta).$$

(81.) If we denote integral (78.) by p, and (80.) by q, we have at once

$$\int_0^\pi \frac{d\theta\log_\epsilon(1+a^3\cos^3\theta)}{1+2m\cos\theta+m^2}=\frac{p-2q}{1-m^2}.$$

More generally we may obtain

$$(82.)\ \int_1^\omega \frac{x^{2r+1}+a^{2r+1}}{x^{2r+1}+a^{2r+1}} \frac{dx}{\sqrt{x^r-1}}. \qquad (83.)\ \int_0^\pi \frac{\cos r\theta\ .\ d\theta}{1+a^r \cos^r\theta}.$$

$$(84.)\ \int_0^\pi d\theta \log_\epsilon (1+a^r \cos^r\theta). \qquad (85.)\ \int_0^\pi \frac{d\theta \log_\epsilon (1+a^r \cos^r\theta)}{1+2m\cos\theta+m^r}.$$

and many others.

II. "On the Reversal of the Lines of Metallic Vapours." By G. D. LIVEING, M.A., Professor of Chemistry, and J. DEWAR, M.A., F.R.S., Jacksonian Professor, University of Cambridge. No. I.

Since the celebrated paper by Kirchhoff, "On the relation between the radiating and absorbing powers of different bodies for light and heat," in which he detailed the remarkable experiments of reversing the lines of lithium and sodium by sunlight and by the vapours of those metals in the flame of a Bunsen's burner, and mentioned the reversal of the brighter lines of potassium, calcium, strontium and barium when the deflagration of the chlorates with milk-sugar was used instead of the flame of a Bunsen's burner, further researches in the same direction have been made by Cornu, Lockyer, and Roberts. The method adopted by Cornu, which had been previously used by Foucault, is one of great ingenuity, dependent upon so arranging the electric arc that the continuous spectrum of the intensely heated poles is examined through an atmosphere of the metallic vapours volatilized around them. By this means Cornu succeeded in reversing several lines in the spectra of the following metals in addition to those above-mentioned, viz., thallium, lead, silver, aluminum, magnesium, cadmium, zinc, and copper. He observed that in general the reversal began with the least refrangible of a group of lines, and gradually extended to the more refrangible lines of the group; and he drew the conclusion that a very thin layer of vapour was sufficient for the reversal. It may be noted that in almost every case the lines seen by him to be reversed were the more highly refrangible of the lines characteristic of each metal, confirming generally the opinion expressed by Stokes in a letter to Lockyer in the Proceedings of the Royal Society for 1876, in which he introduces for the first time the idea of the persistency of different rays with reference to temperature.

The method adopted by Lockyer in the first instance was to view the electric arc through the vapours of the metals volatilized in a stream of hydrogen in a horizontal iron tube.